

THE BINOMIAL THEOREM AND MOTIVIC CLASSES OF UNIVERSAL QUASI-SPLIT TORI

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ABSTRACT. Taking symmetric powers of varieties can be seen as a functor from the category of varieties to the category of varieties with an action by the symmetric group. We study a corresponding map between the Grothendieck groups of these categories. In particular, we derive a binomial formula and use it to give explicit expressions for the classes of universal quasi-split tori in the equivariant Grothendieck group of varieties.

INTRODUCTION

Given a variety X over some field k and a finite G -set S for a finite group G we may form the power X^S . As a variety this is simply the n -fold cartesian product of X with itself, where n is the cardinality of S . The action of G on S induces an action of G on X^S by permuting the factors. This gives a *power functor*

$$(-)^S: \mathrm{Var}_k \rightarrow G\text{-}\mathrm{Var}_k$$

from the category Var_k of varieties over k to the category $G\text{-}\mathrm{Var}_k$ of G -equivariant varieties over k .

By using an exponential function defined on the Grothendieck group of varieties, we show that the power functor extends to a map between the corresponding Grothendieck rings in a natural way. The construction of the exponential function was originally developed by Bouc in the context of Burnside rings in [Bou92].

Denote the set with n elements with its natural action of the symmetric group Σ_n by $[n]$.

The Binomial Theorem. *The symmetric power functor $(-)^{[n]}$ extends to a map between Grothendieck rings $(-)^{[n]}: K_0(\mathrm{Var}_k) \rightarrow K_0(\Sigma_n\text{-}\mathrm{Var}_k)$ such that the relation*

$$(x + y)^{[n]} = \sum_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (x^{[i]} \boxtimes y^{[j]})$$

holds for arbitrary elements x, y in $K_0(\mathrm{Var}_k)$. Furthermore, we have the relation $\{E\}^{[n]} = \mathbb{L}^{dn} \{X\}^{[n]}$ for any rank d vector bundle $E \rightarrow X$.

Our main application of this formula is to obtain an explicit expression for the class of the G -variety \mathbb{G}_m^S in the Grothendieck group $K_0(G\text{-}\mathrm{Var}_k)$ of G -equivariant varieties. The Binomial Theorem allows us to reduce the problem of deriving such a formula to combinatorial calculations in the Burnside ring. Our main result is the following:

Theorem 1. *Let G be a finite group and S a finite G -set with n elements. Then the class of \mathbb{G}_m^S is equal to*

$$\sum_{i=0}^n (-1)^i \lambda^i(S) \mathbb{L}^{n-i}$$

in $K_0(G\text{-}\mathrm{Var}_k)$.

Here λ^i denotes the natural lambda ring operations on $K_0(G\text{-Var}_k)$.

The significance of the G -variety \mathbb{G}_m^S is that it can be regarded as a *universal quasi-split torus*. Recall that a *split torus* over a field k is a finite product of n copies of the group \mathbb{G}_m . In general, an algebraic group T over k is a torus if the base change $T_{\bar{k}}$ to an algebraic closure \bar{k} of k is a split torus. We call T *quasi-split* if it is the Weil-restriction of the group \mathbb{G}_m along a map $\text{Spec } L \rightarrow \text{Spec } k$, where L is a separable algebra over k . In particular, the rational points of such a torus is simply the group L^\times of units in L .

Every rank n quasi-split torus can be obtained from $\mathbb{G}_m^{[n]}$ via descent in the following way. Given a separable k -algebra L of degree n , we can form the configuration space $\text{Conf}^n(\text{Spec } L)$, which is a Σ_n -torsor over k . Descent along this torsor gives a functor $\Sigma_n\text{-Var}_k \rightarrow \text{Var}_k$, and L^\times is the image of $\mathbb{G}_m^{[n]}$ under this functor. The functor induces a lambda-ring homomorphism $K_0(\Sigma_n\text{-Var}_k) \rightarrow K_0(\text{Var}_k)$, and we obtain the following result from [Rök11] as a corollary to our main theorem.

Theorem 2 (Rökaeus). *Let L be a finite separable algebra of degree n over the field k . Then the class of the torus L^\times equals*

$$\sum_{i=0}^n (-1)^i \lambda^i(\text{Spec}(L)) \mathbb{L}^{n-i}$$

in $K_0(\text{Var}_k)$.

The true universal nature of $\mathbb{G}_m^{[n]}$ is best understood in the language of algebraic stacks. The stack quotient $[\mathbb{G}_m^{[n]}/\Sigma_n]$ is a group object over the classifying stack $B\Sigma_n$. A Σ_n -torsor over k corresponds to a map $\text{Spec } k \rightarrow B\Sigma_n$, and L^\times is simply the pull-back of $[\mathbb{G}_m^{[n]}/\Sigma_n]$ along the map corresponding to $\text{Conf}^n(\text{Spec } L)$. Since the category of G -equivariant varieties is equivalent to the category of varieties over the base BG , we can reformulate Theorem 1 as follows:

Theorem 3. *Let G be a finite group and S a finite G -set with n elements. Then the class of $[\mathbb{G}_m^S/G]$ is equal to*

$$\sum_{i=0}^n (-1)^i \lambda^i([S/G]) \mathbb{L}^{n-i}$$

in $K_0(\text{Var}_{BG})$.

The need for explicit expressions for classes of universal quasi-split tori arose in [Ber14], where I computed the classes of certain classifying stacks in the Grothendieck group of stacks. It turns out that classifying stacks of monomial matrices are intimately related to such tori. In this article, the bigger class of stably rational tori is also studied.

It should be pointed out that the proof of Theorem 2 by Rökaeus could easily be adapted to the universal setting. However, using the Binomial Theorem has its merits in its conceptual clearness. For instance, the method could also be used to compute the class of the Weil restriction along a finite étale map of any variety whose class can be expressed as a polynomial in \mathbb{L} . It should also be pointed out that in this article we use a considerably down-powered version of Bouc's machinery. A remark about this concludes the last section. As a final remark, it should be said that the choice of working over a field is done purely for psychological reasons; we could equally well have worked in the relative setting over an arbitrary scheme or algebraic stack.

The rest of the article is organised into three sections. In the first, we review some basic facts about equivariant Grothendieck rings of varieties and Burnside rings. In the second, we construct the exponential function and prove the Binomial

Theorem. In the last, we relate the exponential function to the corresponding construction for Burnside rings and give the proof of Theorem 1.

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PRELIMINARIES

By *variety* in this paper, we will always mean a finitely presented algebraic space over a field k , and we denote the category of such object by Var_k . Given a finite group G , we denote by $G\text{-Var}_k$ the category whose objects are varieties endowed with a G -action and whose morphisms are G -equivariant maps.

Equivariant Grothendieck rings. Given a finite group G , we define the *Grothendieck group of G -equivariant varieties* $K_0(G\text{-Var}_k)$ as the free group on the isomorphism classes $\{X\}$ of objects X in $G\text{-Var}_k$, subject to the following relations:

R1 If $Z \subset X$ is a closed, G -invariant subvariety, then $\{X\} = \{X \setminus Z\} + \{Z\}$.

R2 If $E \rightarrow X$ is a G -equivariant vector bundle of rank d , then $\{E\} = \{\mathbb{A}^d \times X\}$.

The class of the affine line \mathbb{A}_k^1 with the trivial G -action is called the *Lefschetz class* and is denoted by \mathbb{L} .

The group $K_0(G\text{-Var}_k)$ has a structure of commutative ring with identity induced by the categorical product in $G\text{-Var}_k$. We sometimes call $K_0(G\text{-Var}_k)$ the *G -equivariant Grothendieck ring* if we want to emphasise its multiplicative structure. We have an obvious ring homomorphism from $\mathbb{Z}[\mathbb{L}]$ to $K_0(G\text{-Var}_k)$, which turns out to be injective, and sometimes, we prefer to think of the group $K_0(G\text{-Var}_k)$ as a $\mathbb{Z}[\mathbb{L}]$ -module.

The usual Grothendieck ring of varieties, which we denote by $K_0(\text{Var}_k)$, is recovered from the definition above if we take G to be the trivial group. In this case, the relation R2 is redundant.

Equivariant Grothendieck rings for different Grothendieck groups are related to each other by various operations. Given finite groups G and H , we have a functor $\boxtimes: G\text{-Var}_k \times H\text{-Var}_k \rightarrow G \times H\text{-Var}_k$ taking a pair (X, Y) to $X \times Y$ endowed with its natural $G \times H$ -action. This functor induces a corresponding *outer product*

$$\boxtimes: K_0(G\text{-Var}_k) \times K_0(H\text{-Var}_k) \rightarrow K_0(G \times H\text{-Var}_k)$$

on the Grothendieck groups. It is easy to see that this operation is $\mathbb{Z}[\mathbb{L}]$ -bilinear.

Assume that we have an injective group homomorphism $H \rightarrow G$. Then there is natural functor $G\text{-Var}_k \rightarrow H\text{-Var}_k$ given by restricting the group action along $H \rightarrow G$. This functor has a left adjoint given by induction, and the adjoint pair induces a pair of maps

$$\text{Res}_H^G: K_0(G\text{-Var}_k) \rightarrow K_0(H\text{-Var}_k), \quad \text{Ind}_H^G: K_0(H\text{-Var}_k) \rightarrow K_0(G\text{-Var}_k)$$

between the corresponding equivariant Grothendieck rings. The map Res_H^G is a ring homomorphism and Ind_H^G is a group homomorphism. In addition, the map Ind_H^G is $K_0(G\text{-Var}_k)$ -linear in the sense that it satisfies the *projection formula*

$$y \cdot \text{Ind}_H^G(x) = \text{Ind}_H^G(\text{Res}_H^G(y) \cdot x)$$

for $x \in K_0(H\text{-Var}_k)$ and $y \in K_0(G\text{-Var}_k)$. By applying this to the class \mathbb{L} , we see that Ind_H^G is a $\mathbb{Z}[\mathbb{L}]$ -module homomorphism.

The equivariant Grothendieck rings are also endowed with lambda ring structures, and the restriction maps are compatible with these structures. In this article, we will only need the lambda operations for classes which come from the Burnside ring, as will be described below.

The equivariant Grothendieck ring $K_0(\text{Var}_k)$ can also be thought of as the Grothendieck ring for the category Var_{BG} . This is the category of algebraic stacks over the classifying stack BG whose structure morphisms are representable by finitely presented algebraic spaces. In this context, the maps Res_H^G and Ind_H^G are induced by the functors f^* and $f_!$, where $f: BH \rightarrow BG$ is the map between the classifying stacks induced by the inclusion of groups. The functor f^* is simply pull-back along f , and $f_!$ is obtained by post composition with f . Although conceptually appealing, this point of view will not be used in the rest of the exposition.

Remark. The admittedly somewhat unusual use of the term variety requires some explanation. On one hand, the class of algebraic spaces are much better behaved with respect to quotients by finite groups than traditional varieties. For instance, the functor $G\text{-Var}_k \rightarrow \text{Var}_k$ induced by descent along a G -torsor, which was mentioned in the introduction, does not exist for classical varieties. On the other hand, the rings $K_0(\text{Var}_k)$ which we get if we use either the classical definition of variety, or our definition, are canonically isomorphic. Hence there seems to be little motivation for using the term *Grothendieck group of algebraic spaces* rather than the much more well-established *Grothendieck group of varieties*.

If one prefers to work with classical varieties, one can follow the approach used by Heinloth-Bittner in [Bit04], and define a G -variety as a classical variety endowed with a *good* G -action. The results in this article regarding the class of universal quasi-split tori would be equally valid in that setting.

Burnside rings. The *Burnside ring* $A(G)$ for a finite group G is defined as the Grothendieck ring of the monoid of isomorphism classes of finite G -sets. The multiplication and addition in this ring are induced by the categorical product and co-product respectively. The ring $A(G)$ is also endowed with naturally defined lambda operators λ^i , giving it the structure of a (non-special) lambda ring. For an introduction on the Burnside rings and its lambda structure, see [Knu73].

Just as with the equivariant Grothendieck rings, we can relate Burnside rings for different groups G and H with the bilinear operation $\boxtimes: A(G) \times A(H) \rightarrow A(G \times H)$. Given an injective group homomorphism $H \rightarrow G$, we also have the maps Res_H^G and Ind_H^G with the former being a ring homomorphism and the latter being a group homomorphism. The pair also satisfies the projection formula.

We will use the Burnside rings for computing certain coefficients in $K_0(\text{Var}_k)$. There is a natural ring homomorphism $A(G) \rightarrow K_0(G\text{-Var}_k)$ which is induced by the functor taking a G -set to the same G -set viewed as a G -variety. The ring homomorphism respects the lambda structure, and is compatible with the operations \boxtimes , Ind_H^G and Res_H^G in the obvious sense.

As a computational aid when performing calculations in the Burnside ring, we use the *Lefschetz invariant*. A self contained introduction to these techniques is given in [Bou00, §4], and will follow the notation used in this source. The Lefschetz invariant is defined on the class of G -posets. By a G -poset, we mean a finite partially ordered set endowed with a G -action. The G -action is required to respect the ordering in the sense that $gx \leq gy$ for all related pairs $x \leq y$ in the poset and all group elements $g \in G$. Given a G -poset P , we denote the set of i -chains $a_0 < \dots < a_i$ by Sd_i . This set has a natural G -action, and we can consider its class in the Burnside ring, which we also denote by Sd_i . The *Lefschetz invariant* of a G -poset P is defined as the element

$$\Lambda_P = \sum_{i \geq 0} (-1)^i \text{Sd}_i P$$

in $A(G)$. Sometimes, it is more convenient to use the *reduced Lefschetz invariant*, which is defined as $\tilde{\Lambda}_P = \Lambda_P - 1$, where 1 denotes the class of the trivial

G -representation. A fundamental fact is that every element in $A(G)$ can be represented as Λ_P for some G -poset P [Bou92, Lemme 2]. In particular, we have $\Lambda_S = \{S\}$ for every G -set S , where we on the left hand side regard S as a discrete G -poset.

A morphism between two G -posets is defined as a G -equivariant, order preserving map. There is a concept of *homotopy* for such morphisms. We say that the morphisms $f, g: P \rightarrow Q$ are *homotopic* if they are comparable in the pointwise ordering of functions. This gives a notion of *homotopy equivalence* of G -posets. We have the relation $\Lambda_P = \Lambda_Q$ in the Burnside ring provided that P and Q are homotopy equivalent. In particular, the reduced Lefschetz invariant for a poset P vanishes if P has a maximal or minimal element. Indeed, this would imply that P is homotopy equivalent with the G -poset with one element.

For lack of reference, we include a proof of the following basic fact.

Lemma 4. *Let $H \subset G$ be an inclusion of finite groups, and let P be an H -poset. Then we have the equality $\text{Ind}_H^G \Lambda_P = \Lambda_{\text{Ind}_H^G P}$ in $A(G)$.*

Proof. The G -poset $\text{Ind}_H^G P$ consists of the set of equivalence classes of pairs (g, x) with $g \in G$ and $x \in P$. The ordering on $\text{Ind}_H^G P$ is given by the relations $(g, x) \leq (g, x')$ for all $g \in G$ and all relations $x \leq x'$ in P . The equivalence class containing (g, x) is the set of elements on the form $(gh, h^{-1}x)$ for $h \in H$. Fix a natural number i . The G -set $\text{Ind}_H^G \text{Sd}_i P$ is the set of equivalence classes of pairs $(g, x_0 < \dots < x_i)$, with the equivalence relation defined in the obvious way. We have a natural G -equivariant map

$$\eta_i: \text{Ind}_H^G \text{Sd}_i P \rightarrow \text{Sd}_i \text{Ind}_H^G P$$

taking an element $(g, x_0 < \dots < x_i)$ to the chain $(g, x_0) < \dots < (g, x_i)$. It is clear that η_i is injective. Furthermore, any chain $(g_0, x_0) < \dots < (g_i, x_i)$ can be written on the equivalent form $(g_0, x_0) < (g_0, x'_1) < \dots < (g_0, x'_i)$, which proves that it lifts to an element in $\text{Ind}_H^G \text{Sd}_i P$. Hence η_i is an isomorphism, which proves the lemma. \square

THE EXPONENTIAL FUNCTION

Inspired by the corresponding construction for Burnside rings [Bou92], we introduce an exponential function on the Grothendieck ring of varieties. We define the group

$$\mathcal{K}_0(\text{Var}_k) = \prod_{i \geq 0} K_0(\Sigma_i\text{-Var}_k).$$

The elements of $\mathcal{K}_0(\text{Var}_k)$ are denoted as formal power series $\sum_{i \geq 0} a_i T^i$ in the symbol T . Note that each of the coefficients a_i lie in a different group $K_0(\Sigma_i\text{-Var}_k)$. The group $\mathcal{K}_0(\text{Var}_k)$ is given a multiplicative structure, via the binary operation $*$, which is defined by

$$\left(\sum_{i \geq 0} a_i T^i \right) * \left(\sum_{i \geq 0} b_i T^i \right) = \sum_{n \geq 0} \left(\sum_{i+j=n} \text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (a_i \boxtimes b_j) \right) T^n.$$

This gives $\mathcal{K}_0(\text{Var}_k)$ the structure of a ring. We let $\mathcal{U}_0(\text{Var}_k)$ denote the subset of $\mathcal{K}_0(\text{Var}_k)$ of elements of the form $\sum_{i \geq 0} a_i T^i$ with $a_0 = 1$.

Proposition 5. *The operation $*$ on the group $\mathcal{K}_0(\text{Var}_k)$ gives $\mathcal{K}_0(\text{Var}_k)$ the structure of a commutative ring with identity. The identity is given by $1 \in K_0(\text{Var}_k)$. Moreover, the subset $\mathcal{U}_0(\text{Var}_k) \subset \mathcal{K}_0(\text{Var}_k)$ is a group under the operation $*$.*

Proof. Recall that the operation Ind_H^G is linear for any inclusion $H \rightarrow G$ of groups, and that the operation \boxtimes is bilinear and symmetric. From this, it follows easily that the operation $*$ is commutative and distributes over addition. Next, we note that we have the identities

$$\text{Ind}_{\Sigma_i \times \Sigma_{j+l}}^{\Sigma_n} \left(a \boxtimes \text{Ind}_{\Sigma_j \times \Sigma_l}^{\Sigma_{j+l}} (b \boxtimes c) \right) = \text{Ind}_{\Sigma_i \times \Sigma_j \times \Sigma_l}^{\Sigma_n} (a \boxtimes b \boxtimes c)$$

for $a \in K_0(\Sigma_i\text{-Var}_k)$, $b \in K_0(\Sigma_j\text{-Var}_k)$ and $c \in K_0(\Sigma_l\text{-Var}_k)$ with $i + j + l = n$. From this and the symmetry of \boxtimes , it follows that the product of the three elements $\sum a_i T^i$, $\sum b_i T^i$ and $\sum c_i T^i$ can be written

$$\sum_{n \geq 0} \left(\sum_{i+j+l=n} \text{Ind}_{\Sigma_i \times \Sigma_j \times \Sigma_l}^{\Sigma_n} (a_i \boxtimes b_j \boxtimes c_l) \right) T^n$$

regardless of the grouping of the parentheses, which proves associativity. That $1 \in K_0(\text{Var}_k)$ is the identity, follows from the trivial identity $\text{Ind}_{\Sigma_0 \times \Sigma_n}^{\Sigma_n} (1 \boxtimes a) = a$ for $a \in K_0(\Sigma_n\text{-Var}_k)$.

The set $\mathcal{U}_0(\text{Var}_k)$ is clearly closed under multiplication. Let $\sum a_i T^i$ be an element of $\mathcal{K}_0(\text{Var}_k)$ with $a_0 = 1$. We wish to construct an inverse $\sum b_j T^j$ with $b_0 = 1$. But this can be done recursively, in exactly the same way as for usual power series, by defining b_n for each $n > 0$ such that the expression

$$\sum_{i+j=n} \text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (a_i \boxtimes b_j)$$

vanishes. Note that this can be done since we have $\text{Ind}_{\Sigma_0 \times \Sigma_n}^{\Sigma_n} (a_0 \boxtimes b_n) = b_n$ for all $n > 0$ by the assumption on a_0 , which allows us to solve for b_n in each step of the recursion. \square

Next, we explore the power operations which were mentioned in the introduction. Let G be a finite group and S a finite G -set with n elements. Recall that we defined the G -variety X^S as the product of n copies of X together with the natural G -action given by permuting the factors of the product. We wish to examine its class $\{X^S\}$ in the equivariant Grothendieck ring $K_0(G\text{-Var}_k)$. First we note that X^S can be obtained from the variety $X^{[n]}$ via restriction along the group homomorphism $G \rightarrow \Sigma_n$ corresponding to the action on S . Hence it is enough to consider the power operations for the Σ_n -sets $[n]$ for various natural numbers n . Before we continue, we need some basic results on powers of closed immersions and vector bundles.

Proposition 6. *Let Z be a closed subvariety of X in Var_k . Fix a natural number n , and let X_i denote the Σ_n -invariant closed subvariety of $X^{[n]}$ defined by requiring at least $j = n - i$ of the coordinates to lie in Z . Then the sequence*

$$\emptyset = X_{-1} \subset X^{[n]} = X_0 \subset \cdots \subset X_n = X^{[n]}$$

is a filtration of closed immersions such that the stratum $X_i \setminus X_{i-1}$ is isomorphic to $\text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X \setminus Z)^{[i]} \boxtimes Z^{[j]}$ for $0 \leq i \leq n$. In particular, we have the relation

$$\{X^{[n]}\} = \sum_{i+j=n} \text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \left(\{(X \setminus Z)^{[i]}\} \boxtimes \{Z^{[j]}\} \right)$$

in $K_0(\Sigma_n\text{-Var}_k)$.

Proof. By the Yoneda Lemma, we may assume that $Z \subset X$ is an inclusion of sheaves, and by standard arguments, we reduce to the case when $Z \subset X$ is an

inclusion of sets. Consider the following diagram:

$$\begin{array}{ccc} Z^{[i]} \boxtimes (X \setminus Z)^{[j]} & \xrightarrow{\eta} & \text{Res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} X_i \setminus X_{i-1} \\ & \searrow f & \downarrow \text{Res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} f^\natural \\ & & \text{Res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} S \end{array}$$

Here η is the $\Sigma_i \times \Sigma_j$ -equivariant inclusion obtained by restricting the identity map on $X^{[n]}$, and f is an arbitrary $\Sigma_i \times \Sigma_j$ -equivariant function to an arbitrary Σ_n -set S . By the universal property of induction, it is enough to show that there exists a unique Σ_n -equivariant function f^\natural fitting into the diagram. As a candidate, we take f^\natural to be the function defined by

$$x \mapsto \tau f(\tau^{-1}x)$$

where $\tau \in \Sigma_n$ is a permutation such that the first i coordinates of $\tau^{-1}x$ lie in Z . The permutation τ is unique up to multiplication from the right by an element in $\Sigma_i \times \Sigma_j$, so f^\natural is well-defined by $\Sigma_i \times \Sigma_j$ -equivariance of f . Furthermore, the identities

$$f^\natural(\sigma x) = \sigma \tau f(\tau^{-1} \sigma^{-1} \sigma x) = \sigma f^\natural(x),$$

where $\sigma \in \Sigma_n$ is arbitrary and τ is chosen as above, show that f^\natural is Σ_n -equivariant. Since the image of η generates $X_i \setminus X_{i-1}$ as a Σ_n -set, the map f^\natural is also unique with respect to the desired property, which establishes the isomorphism. The statement about the equivariant Grothendieck ring follows by summing over the strata. \square

Instead of just studying the operation $(-)^{[n]}$ for one fixed n , it is beneficial to study the operations for all n simultaneously. We do this by using the ring $\mathcal{K}_0(\text{Var}_k)$. Given a variety X , we consider the element $\sum \{X^{[i]}\} T^i$ in $\mathcal{K}_0(\text{Var}_k)$. Since $\{X^{[0]}\} = 1$, this element lies in $\mathcal{U}_0(\text{Var}_k)$. Let $Z \subset X$ be a closed subvariety of X . By Proposition 6 and the definition of multiplication in the group $\mathcal{U}_0(\text{Var}_k)$, we get the identity

$$\sum_{n \geq 0} \{X^{[n]}\} T^n = \left(\sum_{i \geq 0} \{(X \setminus Z)^{[i]}\} T^i \right) * \left(\sum_{j \geq 0} \{Z^{[j]}\} T^j \right).$$

Hence the association $X \mapsto \sum \{X^{[i]}\} T^i$ respects the defining relations for the Grothendieck group of varieties, and we get a unique group homomorphism from $K_0(\text{Var}_k)$ to $\mathcal{U}_0(\text{Var}_k)$ satisfying $\{X\} \mapsto \sum \{X^{[i]}\} T^i$.

Definition. The group homomorphism described above is called the *exponential function* and is denoted by $\exp: K_0(\text{Var}_k) \rightarrow \mathcal{U}_0(\text{Var}_k)$.

The exponential function gives us a natural way to extend the power operation to the Grothendieck ring $K_0(\text{Var}_k)$.

Definition. Let $x \in K_0(\text{Var}_k)$, and let G be a finite group acting on a set S with n elements. The *power of x by S* is defined as the element

$$x^S := \text{Res}_G^{\Sigma_n} c_n$$

in $K_0(G\text{-Var}_k)$, where c_n denotes the coefficient for T^n in $\exp(x)$.

The results of this section prove the Binomial Theorem stated in the introduction. Indeed, the formula for addition follows directly from the definition of multiplication in the ring $\mathcal{K}_0(\text{Var}_k)$. The statement about vector bundles is the following proposition.

Proposition 7. *Let $E \rightarrow X$ be a vector bundle of rank d in Var_k . Then $E^{[n]} \rightarrow X^{[n]}$ is a vector bundle of rank nd in $\Sigma_n\text{-Var}_k$. In particular, we have $\{E^{[n]}\} = \mathbb{L}^{nd} \cdot \{X^{[n]}\}$ in $K_0(\Sigma_n\text{-Var}_k)$.*

Proof. The fact that $E^{[n]} \rightarrow X^{[n]}$ is a rank nd vector bundle is well-known and easy to prove. See for instance [Eke09, Lemma 2.4] for a proof in the more general context of algebraic stacks. The statement about the class in the equivariant Grothendieck ring is simply the defining relation R2. \square

CLASSES OF QUASI-SPLIT TORI

The methods introduced in the previous section give us a convenient way to compute the class of a quasi-split torus in the Grothendieck ring of varieties. As explained in the introduction, it is enough to consider the universal case.

Proposition 8. *The class of the universal quasi-split torus $\mathbb{G}_m^{[n]}$ of rank n is given by the expression*

$$\sum_{i=0}^n \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \left((-1)^{[i]} \boxtimes 1 \right) \cdot \mathbb{L}^{n-i}$$

in the equivariant Grothendieck ring $K_0(\Sigma_n\text{-Var}_k)$.

Proof. Since the class of \mathbb{G}_m is $\mathbb{L} - 1$ in $K_0(\text{Var}_k)$, the class of $\mathbb{G}_m^{[n]}$ is given by

$$\sum_{i=0}^n \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \left((-1)^{[i]} \boxtimes \mathbb{L}^{[n-i]} \right)$$

according to the Binomial Theorem. From Proposition 7, it follows that $\mathbb{L}^{[n-i]}$ equals \mathbb{L}^{n-i} . Hence the result follows from the $\mathbb{Z}[\mathbb{L}]$ -linearity of the operations \boxtimes and $\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n}$. \square

Theorem 1 and its corollaries will now follow provided that we establish the identity

$$(-1)^i \lambda^i([n]) = \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \left((-1)^{[i]} \boxtimes 1 \right)$$

for the coefficients in the expression for the class of $\mathbb{G}_m^{[n]}$ given in Proposition 8. This computation can be done entirely in the Burnside ring, which allows us to use the methods introduced by Bouc in [Bou92]. We briefly recapitulate the parts of the theory that we need.

The ring $\mathcal{K}_0(\text{Var}_k)$ has an exact analog for Burnside rings, namely the ring

$$\mathcal{A} = \prod_{i \geq 0} A(\Sigma_i).$$

We think of elements of \mathcal{A} as formal power series $\sum a_i T^i$ and define the multiplication using the same formula as for $\mathcal{K}_0(\text{Var}_k)$. We also have an exponential function $\exp: \mathbb{Z} \rightarrow \mathcal{A}$ induced by taking a finite set S with n elements to the power series $\sum \{S^{[i]}\} T^i$. This operation extends to all of \mathbb{Z} .

The group homomorphisms $A(\Sigma_i) \rightarrow K_0(\Sigma_i\text{-Var}_k)$ of the coefficient rings combine to a group homomorphism $\iota: \mathcal{A} \rightarrow \mathcal{K}_0(\text{Var}_k)$. Since the induction and outer products commute with the maps $A(\Sigma_i) \rightarrow K_0(\Sigma_i\text{-Var}_k)$, we see that ι is in fact a ring homomorphism. It also respects the exponential functions, which shows that we can compute the powers $n^{[i]}$ in the Burnside rings for all integers n and then map the result to $K_0(\Sigma_i\text{-Var}_k)$.

Powers in the Burnside rings can be computed by means of the Lefschetz invariant. Given a poset P and a natural number i , we define $P^{[i]}$ as the i -fold product of P with itself. This set has a structure of Σ_i -poset, with the Σ_i -action permuting the factors and the ordering given by coordinate-wise comparison. More specifically,

we have $(x_1, \dots, x_i) \leq (x'_1, \dots, x'_i)$ provided that $x_j \leq x'_j$ for $1 \leq j \leq i$. Given an integer n and a poset P such that $n = \Lambda_P$, we have $n^{[i]} = \Lambda_{P^{[i]}}$ for all natural numbers i .

In order to compute the powers of -1 , we introduce the following notation. Let G be a finite group and S a finite G -set. Given an integer i , we let $\Omega_{\leq i} S$ denote the set of non-empty subsets of S with cardinality less or equal to i . The set $\Omega_{\leq i} S$ has G -poset structure with its G -action inherited from S in the obvious way, and the ordering given by inclusion. We use the symbol S_+ to denote the disjoint union of the G -set S with the one-point G -set $\{\bullet\}$.

Lemma 9. *We have the identity $(-1)^{[n]} = \tilde{\Lambda}_{\Omega_{\leq n}([n]_+)}$ in $K_0(\Sigma_n\text{-Var}_k)$ for each natural number n .*

Proof. By [Bou92, Lemme 4], we have the identity $(-1)^{[n]} = -\tilde{\Lambda}_{2^{*n}}$. The Σ_n -poset 2^{*n} can be described as the set F of functions from $[n]$ to $\{a, b, \infty\}$ which are not constant ∞ . The Σ_n -action is given by $(\sigma \cdot \varphi)(x) = \varphi(\sigma^{-1}x)$ for $\sigma \in \Sigma_n$. The set $\{a, b, \infty\}$ has an ordering given by $a < \infty$, $b < \infty$, and the ordering on the set F is given by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [n]$. This gives F a Σ_n -poset structure. Consider the map $f: F \rightarrow \Omega_{\leq n}([n]_+)$ defined by

$$\varphi \mapsto \begin{cases} \varphi^{-1}(a) & \text{if } b \notin \text{Im } \varphi, \\ \varphi^{-1}(a) \cup \{\bullet\} & \text{otherwise.} \end{cases}$$

We also have a map g in the other direction, which takes a subset U to the function φ_U defined by

$$\varphi_U(x) = \begin{cases} a & \text{if } x \in U, \\ b & \text{if } x \notin U \text{ and } \bullet \in U, \\ \infty & \text{otherwise.} \end{cases}$$

Both the maps f and g are Σ_n -equivariant and order-reversing. One easily verifies that $h \circ g$ is the identity and that $g \circ h$ is comparable with the identity. Hence $\Omega_{\leq n}([n]_+)$ is homotopy equivalent to F considered with its opposite ordering. Since reversing the ordering of a G -poset clearly does not change its Lefschetz invariant, the statement in the lemma follows. \square

Next, we need to investigate how the G -posets $\Omega_{\leq i}(S)$ transform under induction. Fix natural numbers $i' \leq i$. We let $\Omega_{\leq i', i}(S)$ denote the set of flags $U \subset V$ of non-empty subsets of S , where U has at most i' elements and V has exactly i elements. These flags are ordered by level-wise inclusion, and we have a natural action of G on $\Omega_{\leq i', i}(S)$ respecting the ordering.

Lemma 10. *Let $i' \leq i \leq n$ be natural numbers and define $j = n - i$. We consider $\Omega_{\leq i'}([i])$ a $\Sigma_i \times \Sigma_j$ -poset by letting the second factor act trivially. Then*

$$\text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \Omega_{\leq i'}([i]) \simeq \Omega_{\leq i', i}([n])$$

as Σ_n -posets.

Proof. We use a similar argument as in the proof of Proposition 6. Consider the $\Sigma_i \times \Sigma_j$ -poset morphism

$$\eta: \Omega_{\leq i'}([i]) \rightarrow \text{Res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \Omega_{\leq i', i}([n])$$

that takes a subset U of $[i]$ to the flag $U \subset [i]$. We prove that any $\Sigma_i \times \Sigma_j$ -poset morphism $f: \Omega_{\leq i'}([i]) \rightarrow \text{Res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} P$, where P is an arbitrary Σ_n -poset, uniquely extends to a Σ_n -poset morphism $f^\natural: \Omega_{\leq i', i}([n]) \rightarrow P$ via η . As a candidate, we take the function

$$f^\natural(U \subset V) = \tau f(\tau^{-1}U)$$

where τ is a permutation such that $\tau^{-1}V = [i]$. The permutation τ is unique up to right multiplication by a permutation in $\Sigma_i \times \Sigma_j$. Hence f^\natural is well-defined by $\Sigma_i \times \Sigma_j$ -equivariance of f . For an arbitrary element $\sigma \in \Sigma_n$, we have the equalities

$$\sigma f^\natural(U \subset V) = \sigma \tau f(\tau^{-1}U) = \sigma \tau f((\tau^{-1}\sigma^{-1})\sigma U) = f^\natural(\sigma(U \subset V)).$$

Here the last step follows from the identity $(\tau^{-1}\sigma^{-1})\sigma V = [i]$. Since f^\natural clearly is order-preserving, this shows that f^\natural is a Σ_n -poset morphism. Finally, since the image of η generates $\Omega_{\leq i', i}([n])$ as a Σ_n -set, the extension f^\natural is a unique with respect to the desired property, which concludes the proof. \square

Finally, we need an expression for the lambda operations in terms of Lefschetz invariants of equivariant posets. Such expressions have been computed by Bouc and Rökaeus in [BR09]. We state the result together with an auxiliary result from the same article.

Lemma 11 (Bouc–Rökaeus). *Let G be a finite group and S a finite G -set. Then we have the following identities in the Burnside ring $A(G)$:*

- (a) $\lambda^n(S) = (-1)^{n-1} \tilde{\Lambda}_{\Omega_{\leq n}(S_+)}$
- (b) $\tilde{\Lambda}_{\Omega_{\leq n}(S_+)} = \tilde{\Lambda}_{\Omega_{\leq n}(S)} - \tilde{\Lambda}_{\Omega_{\leq n-1}(S)}$

Now we are ready to combine the results and prove the main theorem about the expression for the class of the universal quasi-split torus.

Proof of Theorem 1. By Lemma 11 (b), we have the identities

$$\tilde{\Lambda}_{\Omega_{\leq i}([i]_+)} = \tilde{\Lambda}_{\Omega_{\leq i}([i])} - \tilde{\Lambda}_{\Omega_{\leq i-1}([i])} = \Lambda_{\Omega_{\leq i}([i])} - \Lambda_{\Omega_{\leq i-1}([i])}.$$

Using that taking the Lefschetz invariant commutes with induction as proved in Lemma 4, together with the explicit description from Lemma 10 of the induced posets involved, we get

$$\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \tilde{\Lambda}_{\Omega_{\leq i}([i]_+)} = \Lambda_{\Omega_{\leq i, i}([n])} - \Lambda_{\Omega_{\leq i-1, i}([n])}.$$

Next, we note that the element $\text{Sd}_j \Omega_{\leq i, i}([n]) - \text{Sd}_j \Omega_{\leq i-1, i}([n])$ in $A(\Sigma_n)$ is effective. It is represented by the Σ_n -set of length j chains of flags $V \subset U$ in $[n]$, with V non-empty and U having cardinality i , whose maximal element satisfies $V = U$. But it is obvious that the set U is redundant in this description. The set is equivariantly isomorphic to the set of chains of non-empty subsets of $[n]$ with the maximal subset of cardinality i , which has the class $\text{Sd}_j \Omega_{\leq i}([n]) - \text{Sd}_j \Omega_{\leq i-1}([n])$. By using Lemma 11 (b) again, we therefore get

$$\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \tilde{\Lambda}_{\Omega_{\leq i}([i]_+)} = \Lambda_{\Omega_{\leq i}([n])} - \Lambda_{\Omega_{\leq i-1}([n])} = \tilde{\Lambda}_{\Omega_{\leq i}([n]_+)}.$$

By using the expression $(-1)^{[i]} = -\tilde{\Lambda}_{\Omega_{\leq i}([i]_+)}$ derived in Lemma 9, we get

$$\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \left((-1)^{[i]} \boxtimes 1 \right) = -\tilde{\Lambda}_{\Omega_{\leq i}([n]_+)}.$$

By applying Lemma 11 (a) to the right hand side of this identity, we see that the coefficients in the formula of Proposition 8 coincides with the coefficients in Theorem 1. \square

Remark. Note that we have only used a very special case of Bouc’s construction of the exponential function. He considers the more general ring $\mathcal{A}(G) = \prod_{i \geq 0} A(G \wr \Sigma_i)$ where G is an arbitrary finite group. Here $G \wr \Sigma_i$ denotes the wreath product of G by Σ_i . In particular, there are power operations $(-)^{[i]}: A(G) \rightarrow A(G \wr \Sigma_i)$ for all $i \in \mathbb{N}$. A similar construction could be made for equivariant Grothendieck rings of varieties. In this setting, one could compute the class of the Weil restriction along a finite étale map for more general varieties X . More precisely, the class $\{X\}$ could be a polynomial in \mathbb{L} with coefficients in finite étale classes.

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